

DEFORMATIONS OF A MATCHED PAIR AND SCHREIER TYPE THEOREMS FOR BICROSSED PRODUCT OF GROUPS

A. L. AGORE AND G. MILITARU

ABSTRACT. We prove that the bicrossed product of two groups is a quotient of the pushout of two semidirect products. A matched pair of groups (H, G, α, β) is deformed using a combinatorial datum (σ, v, r) consisting of an automorphism σ of H , a permutation v of the set G and a transition map $r : G \rightarrow H$ in order to obtain a new matched pair $(H, (G, *), \alpha', \beta')$ such that there exist an σ -invariant isomorphism of groups $H \rtimes_{\alpha \rtimes \beta} G \cong H \rtimes_{\alpha' \rtimes \beta'} (G, *)$. Moreover, if we fix the group H and the automorphism $\sigma \in \text{Aut}(H)$ then any σ -invariant isomorphism $H \rtimes_{\alpha \rtimes \beta} G \cong H \rtimes_{\alpha' \rtimes \beta'} G'$ between two arbitrary bicrossed product of groups is obtained in a unique way by the above deformation method. As applications two Schreier type classification theorems for bicrossed product of groups are given.

INTRODUCTION

The aim of the paper is to bring back to attention and revitalize one of the most famous open problems of group theory formulated in the first half of the last century ([8], [17], [19]). It can be seen as the dual of the more famous *extension problem* of O. L. Hölder and it is called the *factorization problem*. The statement is very simple and tempting:

Let H and G be two given groups. Describe and classify up to an isomorphism all groups E that factorize through H and G : i.e. E contains H and G as subgroups such that $E = HG$ and $H \cap G = 1$.

Leaving aside the classification part introduced above, the first part of the problem was formulated in 1937 by O. Ore [17] but its roots are much older and descend to E. Maillet's 1900 paper [14]. Even though if the statement is very simple, as many famous problems in mathematics are, little progress has been made since then. We dare to say that this one is even more difficult than the more popular extension problem. In the case of two cyclic groups H and G , not both finite, the problem was started by L. Rédei in [19] and finished by P.M. Cohn in [7], without the classification part introduced above. To the best of our knowledge this seems to be the only case where the complete answer is known. If H and G are both finite cyclic groups the problem is more difficult and seems to be still an open question, even though J. Douglas [8] has devoted four papers and over two dozen theorems to the subject. Recently, in [1, Theorem 2.1] the problem was solved

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in the case that one of the finite cyclic groups is of prime order. Using a famous theorem of Frobenius a Schur-Zassenhaus type theorem was proven: any group E that factorizes through two finite cyclic groups, one of them being of prime order, is isomorphic to a semidirect product between the two cyclic groups of the same order.

More popular in group theory was the converse of the factorization problem: given a group E find all *exact factorizations* of it, that is, all subgroups H and G of E such that $E = HG$ and $H \cap G = 1$. Starting with the 1980's various papers dealing with this problem were written (see [4], [9], [18], [22] and their list of references). Derived from this problem is the following: describe and characterize the class of (finite simple) groups that do not admit an exact factorization between two proper subgroups. Having in mind the abelian case such a group will be called an *indecomposable group*: the quaternion group Q , \mathbb{Z}_{p^n} for a prime integer p or the alternating group A_6 are typical examples of indecomposable groups.

An important step related to the factorization problem was the construction of the bicrossed product $H {}_{\alpha} \bowtie_{\beta} G$ associated to a matched pair (H, G, α, β) given by M. Takeuchi [20]: α is a left action of the group G on the set H , β is a right action of the group H on the set G satisfying two compatibility conditions. A group E factorizes through two subgroups H and G if and only if there exists a matched pair (H, G, α, β) such that

$$\theta : H {}_{\alpha} \bowtie_{\beta} G \rightarrow E, \quad \theta(h, g) = hg$$

is an isomorphism of groups. Thus the factorization problem can be restated in a computational manner as follows:

Let H and G be two given groups. Describe all matched pairs (H, G, α, β) and classify up to an isomorphism all bicrossed products $H {}_{\alpha} \bowtie_{\beta} G$.

The motivation for the above problem is triple: first of all, the problem presents an interest in itself in group theory. On the other hand the construction of the bicrossed product provides the easiest way of constructing finite quantum groups [15], hence the classification theorems from group level lead us to classification theorems for finite quantum groups. Finally, the bicrossed product construction at the level of groups served as a model for similar constructions in other fields of mathematics like: algebras [6], coalgebras [5], groupoids [2], Hopf algebras [20], locally compact groups [3] or locally compact quantum groups [21], Lie Algebras [16] or Lie groups [13]. Thus, the above problem can be easily formulated for each of the above different levels where the bicrossed product construction was made. For instance, at the level of algebras (the bicrossed product of two algebras is also called *twisted tensor product algebra*) the first steps were already made in the last years: the story started with [5, Examples 2.11] where all bicrossed product between two group algebras of dimension two are completely described and classified. Recently, the classification of all bicrossed product between the algebras k^2 and k^m was finished in [12] and the description of some bicrossed products between two polynomial algebras $k[X]$ and $k[Y]$ was started in [10]. On the other hand, in [11] only a sufficient condition for the isomorphism between two bicrossed products of algebras that fix one of the algebra is given under the name of *invariance under twisting* problem.

This paper is devoted to the classification part of the factorization problem at the group level. Namely we shall ask the following question: when are two bicrossed products $H \rtimes_{\alpha} G$ and $H \rtimes_{\alpha'} G$ isomorphic? The organization of the paper is the following: in Section 1 we recall the construction of the bicrossed product of two groups given by M. Takeuchi. It is a generalization of the semidirect product construction for the case when none of the factors is required to be normal. The first natural question arises: how far is a bicrossed product from being a semidirect product? Proposition 1.2 gives the first answer to the question: we prove that the bicrossed product of two groups is a quotient of the pushout of two semidirect products over the direct product of the subgroups of invariants of the actions α and β . In Section 2 we start the classification part of the factorization problem. The main result is Theorem 2.2 : for any matched pair of groups (H, G, α, β) and any triple (σ, v, r) , consisting of an automorphism σ of H , a permutation v on the set G and a transition map $r : G \rightarrow H$ satisfying a certain compatibility condition, a new matched pair $(H, (G, *), \alpha', \beta')$ is constructed such that there exists an σ -invariant isomorphism of groups $H \rtimes_{\alpha} G \cong H \rtimes_{\alpha'} (G, *)$. The importance of the result is given by the converse: if we fix the group H and the automorphism $\sigma \in \text{Aut}(H)$, then any σ -invariant isomorphism $H \rtimes_{\alpha} G \cong H \rtimes_{\alpha'} G'$ between two arbitrary bicrossed products of groups is obtained in a unique way by the above deformation method. As applications in Section 3 two Schreier type classification theorems for bicrossed products of groups are given. They are formulated using the language of category theory. Let H and G be two fixed groups: we define a category $B_1(H, G)$ having as object the set of all matched pairs (H, G, α, β) and morphisms are defined as morphisms between two bicrossed products that fix one of the groups. Theorem 3.3 gives a bijection between the set of objects of the skeleton of the category $B_1(H, G)$ and a certain pointed set $K^2(H, G)$ that will play for the classification problem of bicrossed products the same role as the second cohomology group does for the classification of the extension problem. Returning to the question of how far a bicrossed product is from being a semidirect product, Corollary 3.4 and Corollary 3.5 give two necessary and sufficient conditions for a bicrossed product to be isomorphic to a semidirect product of groups in the category $B_1(H, G)$. Theorem 3.9 is the second Schreier type theorem for bicrossed products: this time we fix two groups H, G and $\beta : G \times H \rightarrow G$ a right action of the group H on the set G and the classification theorem is more restrictive than the one given in Theorem 3.3. In the last section we give some examples: we compute and count explicitly the set of all matched pairs $(C_3, C_m, \alpha, \beta)$, where C_m is a cyclic group of order m , and the pointed set $K^2(C_3, C_6)$ constructed in Theorem 3.3 is shown to have three elements.

1. PRELIMINARIES

Let us fix the notation that will be used throughout the paper. Let H and G be two groups and $\alpha : G \times H \rightarrow H$ and $\beta : G \times H \rightarrow G$ two maps. We use the notation

$$\alpha(g, h) = g \triangleright h \quad \text{and} \quad \beta(g, h) = g \triangleleft h$$

for all $g \in G$ and $h \in H$. The map α (resp. β) is called trivial if $g \triangleright h = h$ (resp. $g \triangleleft h = g$) for all $g \in G$ and $h \in H$. We recall that α is an action as automorphism if it is a left action of the group G on the set H and $g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2)$, for all $g \in G, h_1,$

$h_2 \in H$. Similarly, β is an action as automorphism if it is a right action of the group H on the set G and $(g_1 g_2) \triangleleft h = (g_1 \triangleleft h)(g_2 \triangleleft h)$ for all $g_1, g_2 \in G$ and $h \in H$. $\text{Aut}(H)$ is the group of automorphisms of H and C_n is the cyclic group of order n .

Let H and G be two groups with the multiplications $m_H : H \times H \rightarrow H$, $m_G : G \times G \rightarrow G$, units 1_H and respectively 1_G and $R : G \times H \rightarrow H \times G$ a map. We shall define a new multiplication on the set $H \times G$ using R instead of the usual flip $\tau : G \times H \rightarrow H \times G$, $\tau(g, h) = (h, g)$ as follows:

$$m_{H \times G, R} : H \times G \times H \times G \rightarrow H \times G, \quad m_{H \times G, R} := (m_H \times m_G) \circ (I \times R \times I)$$

Let $\alpha := \pi_1 \circ R : G \times H \rightarrow H$, $\beta := \pi_2 \circ R : G \times H \rightarrow G$, where π_i is the projection on the i -component; we shall denote $\alpha(g, h) = g \triangleright h$ and $\beta(g, h) = g \triangleleft h$, for all $g \in G$ and $h \in H$. Then $R(g, h) = (g \triangleright h, g \triangleleft h)$ and the multiplication $m_{H \times G, R}$ on $H \times G$ can be explicitly written as follows:

$$(h_1, g_1) \cdot_R (h_2, g_2) = (h_1(g_1 \triangleright h_2), (g_1 \triangleleft h_2)g_2) \quad (1)$$

for all $h_1, h_2 \in H$ and $g_1, g_2 \in G$.

It can be easily shown that $(H \times G, m_{H \times G, R})$ is a group with $(1_H, 1_G)$ as a unit if and only if (H, G, α, β) is a *matched pair* in the sense of Takeuchi ([20]): i.e. α is a left action of the group G on the set H , β is a right action of the group H on the set G and the following two compatibility conditions hold:

$$g \triangleright (h_1 h_2) = (g \triangleright h_1)((g \triangleleft h_1) \triangleright h_2) \quad (2)$$

$$(g_1 g_2) \triangleleft h = (g_1 \triangleleft (g_2 \triangleright h))(g_2 \triangleleft h) \quad (3)$$

for all $h, h_1, h_2 \in H$ and $g, g_1, g_2 \in G$. It follows from (2) and (3) that:

$$g \triangleright 1_H = 1_H \quad \text{and} \quad 1_G \triangleleft h = 1_G \quad (4)$$

for all $h \in H$ and $g \in G$.

If (H, G, α, β) is a matched pair, the new group obtained on the set $H \times G$ will be denoted by $H \rtimes_{\alpha \bowtie \beta} G = H \bowtie G$ and will be called the *bicrossed product* (*knit product* or *Zappa-Szép product*) of H and G . We note that $i_H : H \rightarrow H \bowtie G$, $i_H(h) = (h, 1)$ and $i_G : G \rightarrow H \bowtie G$, $i_G(g) = (1, g)$, for all $h \in H$, $g \in G$ are morphisms of groups and hence $H \times \{1\} \cong H$ and $\{1\} \times G \cong G$ are subgroups of $H \bowtie G$. Moreover, every element (h, g) of $H \bowtie G$ can be written uniquely as a product of an element of $H \times \{1\}$ and of an element of $\{1\} \times G$ as follows:

$$(h, g) = (h, 1) \cdot (1, g) \quad (5)$$

Conversely, this observation characterizes the bicrossed product. Let E be a group H , $G \leq E$ be subgroups such that any element of E can be written uniquely as a product of an element of E and an element of G . Then there exists a matched pair (H, G, α, β) such that

$$\theta : H \bowtie G \rightarrow E, \quad \theta(h, g) = hg$$

is a group isomorphism ([20]). The maps α and β play a symmetric role: if (H, G, α, β) is a matched pair then we can construct a new matched pair $(G, H, \tilde{\alpha}, \tilde{\beta})$ such that there exists a canonical isomorphism of groups $H \rtimes_{\alpha \bowtie \beta} G \cong G \rtimes_{\tilde{\alpha} \bowtie \tilde{\beta}} H$ ([1, Proposition 2.5]).

Remark 1.1. Let H and G be two groups and $\beta : G \times H \rightarrow G$ the trivial action. Then (H, G, α, β) is a matched pair if and only if $\alpha : G \times H \rightarrow H$ is an action of G on H as group automorphisms. In this case the bicrossed product $H \bowtie G$ is exactly the left version of the semidirect product $H_\alpha \ltimes G$.

Assume now that the map α is the trivial action. Then (H, G, α, β) is a matched pair if and only if β is a right action of H on G as group automorphisms. In this case the bicrossed product $H \bowtie G$ is exactly the right version of the semidirect product $H \rtimes_\beta G$. It can be easily proved that $H \rtimes_\beta G \cong G_\varphi \ltimes H$, where $\varphi = \varphi_\beta$ is the action of H on G as group automorphisms given by

$$\varphi : H \rightarrow \text{Aut}(G), \quad \varphi(h)(g) = \left(g^{-1} \triangleleft h^{-1}\right)^{-1}$$

for all $h \in H$ and $g \in G$ ([1, Remark 2.6]).

A matched pair (H, G, α, β) is called *proper* if α and β are both nontrivial actions.

The above Remark shows that the semidirect product is a special case of the bicrossed product construction. It is therefore natural to ask the converse: *Can a bicrossed product be obtained from semidirect products of groups?* In what follows we shall give a first answer to this question: a bicrossed product can be obtained as a quotient of a pushout of two semidirect products of groups.

Let (H, G, α, β) be a matched pair and let us denote by $\text{Fix}(H)$ and $\text{Fix}(G)$ the invariants of the two actions α and β :

$$\text{Fix}(H) := \{h \in H \mid g \triangleright h = h, \forall g \in G\}, \quad \text{Fix}(G) := \{g \in G \mid g \triangleleft h = g, \forall h \in H\}$$

Using the compatibility conditions (2) and (3) we shall prove that $\text{Fix}(H)$ is a subgroup of H and $\text{Fix}(G)$ a subgroup of G . Indeed, from (4) we obtain that $1_H \in \text{Fix}(H)$ and for $h_1, h_2 \in \text{Fix}(H)$ we have:

$$g \triangleright (h_1 h_2) \stackrel{(2)}{=} (g \triangleright h_1)((g \triangleleft h_1) \triangleright h_2) = h_1 h_2$$

i.e. $h_1 h_2 \in \text{Fix}(H)$. On the other hand:

$$1_H \stackrel{(4)}{=} g \triangleright 1_H = g \triangleright (h_1^{-1} h_1) \stackrel{(2)}{=} (g \triangleright h_1^{-1})((g \triangleleft h_1^{-1}) \triangleright h_1) = (g \triangleright h_1^{-1}) h_1$$

Thus $g \triangleright h_1^{-1} = h_1^{-1}$, i.e. $h_1^{-1} \in \text{Fix}(H)$. In a similar way we can show that $\text{Fix}(G)$ is a subgroup of G . Using the compatibility condition (2) we obtain that the map given by:

$$\varphi_\triangleright : \text{Fix}(G) \rightarrow \text{Aut}(H), \quad \varphi_\triangleright(g)(h) := g \triangleright h$$

for all $g \in \text{Fix}(G)$, $h \in H$ is a morphism of groups. Thus we can construct the left version of the semidirect product associated to the triple $(H, \text{Fix}(G), \varphi_\triangleright)$: that is $H_{\varphi_\triangleright} \ltimes \text{Fix}(G) := H \times \text{Fix}(G)$ with the multiplication:

$$(h, g)(h', g') = (h(g \triangleright h'), gg')$$

for all $h, h' \in H$ and $g, g' \in \text{Fix}(G)$. Similarly, using (3) we obtain that the map given by:

$$\psi_\triangleleft : \text{Fix}(H) \rightarrow \text{Aut}(G), \quad \psi_\triangleleft(h)(g) := g \triangleleft h$$

for all $h \in \text{Fix}(H)$, $g \in G$ is a morphism of groups and we can construct the right version of the semidirect product associated to the triple $(G, \text{Fix}(H), \psi_{\triangleleft})$: i.e. $\text{Fix}(H) \rtimes_{\psi_{\triangleleft}} G := \text{Fix}(H) \times G$ with the multiplication:

$$(h, g)(h', g') = (hh', (g \triangleleft h')g')$$

for all $h, h' \in \text{Fix}(H)$ and $g, g' \in G$. Moreover, the inclusion maps

$$\bar{i} : \text{Fix}(H) \times \text{Fix}(G) \hookrightarrow H_{\varphi_{\triangleright}} \ltimes \text{Fix}(G) \quad \text{and} \quad \bar{j} : \text{Fix}(H) \times \text{Fix}(G) \hookrightarrow \text{Fix}(H) \rtimes_{\psi_{\triangleleft}} G$$

are morphisms of groups by straightforward verifications.

On the other hand we can easily prove that the canonical inclusions

$$i : H_{\varphi_{\triangleright}} \ltimes \text{Fix}(G) \hookrightarrow H_{\alpha \bowtie \beta} G, \quad i(h, g) = (h, g)$$

and

$$j : \text{Fix}(H) \rtimes_{\psi_{\triangleleft}} G \hookrightarrow H_{\alpha \bowtie \beta} G, \quad j(h, g) = (h, g)$$

are morphisms of groups. Indeed for $h, h' \in H$ and $g, g' \in \text{Fix}(G)$ we have:

$$i(h, g) \cdot i(h', g') = (h(g \triangleright h'), (g \triangleleft h')g') \stackrel{g \in \text{Fix}(G)}{=} (h(g \triangleright h'), gg') = i((h, g)(h', g'))$$

Thus the two semidirect products constructed above, $H_{\varphi_{\triangleright}} \ltimes \text{Fix}(G)$ and $\text{Fix}(H) \rtimes_{\psi_{\triangleleft}} G$, are subgroups of the bicrossed product $H_{\alpha \bowtie \beta} G$. To conclude, we obtained a commutative diagram in the category of groups

$$\begin{array}{ccc} \text{Fix}(H) \times \text{Fix}(G) & \xrightarrow{\bar{j}} & \text{Fix}(H) \rtimes_{\psi} G \\ \downarrow \bar{i} & & \downarrow j \\ H_{\varphi} \ltimes \text{Fix}(G) & \xrightarrow{i} & H_{\alpha \bowtie \beta} G \end{array} \quad (6)$$

Using the construction of the pullback in the category of groups it follows that the pair $(\text{Fix}(H) \times \text{Fix}(G), (\bar{i}, \bar{j}))$ is a pullback of the morphisms $i : H_{\varphi_{\triangleright}} \ltimes \text{Fix}(G) \hookrightarrow H_{\alpha \bowtie \beta} G$ and $j : \text{Fix}(H) \rtimes_{\psi_{\triangleleft}} G \hookrightarrow H_{\alpha \bowtie \beta} G$.

Proposition 1.2. *Let (H, G, α, β) be a matched pair of groups and $(X, (\varphi, \psi))$ be the pushout in the category of groups of the diagram*

$$\begin{array}{ccc} \text{Fix}(H) \times \text{Fix}(G) & \xrightarrow{\bar{j}} & \text{Fix}(H) \rtimes_{\psi} G \\ \downarrow \bar{i} & & \downarrow \varphi \\ H_{\varphi} \ltimes \text{Fix}(G) & \xrightarrow{\psi} & X \end{array}$$

Then the bicrossed product $H_{\alpha \bowtie \beta} G$ is isomorphic to a quotient group of X .

Proof. The diagram (6) is commutative and $(X, (\varphi, \psi))$ is the pushout of the pair (\bar{i}, \bar{j}) : thus there exists an unique morphism of groups $\theta : X \rightarrow H_{\alpha \bowtie \beta} G$ such that $\theta \circ \psi = i$ and $\theta \circ \varphi = j$. Let $(h, g) \in H_{\alpha \bowtie \beta} G$: as $(h, 1_G) \in H_{\varphi_{\triangleright}} \ltimes \text{Fix}(G)$ and $(1_H, g) \in \text{Fix}(H) \rtimes_{\psi_{\triangleleft}} G$ we obtain

$$(h, g) = (h, 1_G)(1_H, g) = i(h, 1_G)j(1_H, g) = \theta(\psi(h, 1_G))\theta(\varphi(1_H, g)) = \theta(\psi(h, 1_G)\varphi(1_H, g))$$

that is θ is surjective. Thus $H_{\alpha \bowtie \beta} G$ is a quotient group of X . \square

We end the section with a problem that can be of interest for a further study:

Let "P" be a property in the category of groups. Give a necessary and sufficient condition such that $H \rtimes_{\alpha\rtimes\beta} G$ has the property "P".

In the following we give an example in the case that "P" is the property of being abelian or cyclic.

Proposition 1.3. *Let (H, G, α, β) be a matched pair of groups. Then:*

(1) *The center of the bicrossed product $H \rtimes_{\alpha\rtimes\beta} G$ is given by:*

$$Z(H \rtimes_{\alpha\rtimes\beta} G) = \{(h, g) \in \text{Fix}(H) \times \text{Fix}(G) \mid g \triangleright x = h^{-1}xh, y \triangleleft h = gyg^{-1}, \forall x \in H, y \in G\}$$

(2) *$H \rtimes_{\alpha\rtimes\beta} G$ is an abelian group if and only if H and G are abelian groups and α and β are the trivial actions;*

(3) *$H \rtimes_{\alpha\rtimes\beta} G$ is a cyclic group if and only if α and β are the trivial actions and H, G are finite cyclic groups of coprime orders.*

Proof. An element $(h, g) \in H \rtimes_{\alpha\rtimes\beta} G$ belongs to the center of the group if and only if $(h, g)(x, 1) = (x, 1)(h, g)$ and $(h, g)(1, y) = (1, y)(h, g)$, for all $x \in H$ and $y \in G$. This is equivalent to $h(g \triangleright x) = xh$, $g \triangleleft x = g$, $y \triangleright h = h$ and $(y \triangleleft h)g = gy$, for all $x \in H$, $y \in G$. Hence $h \in \text{Fix}(H)$, $g \in \text{Fix}(G)$, $g \triangleright x = h^{-1}xh$, $y \triangleleft h = gyg^{-1}$ for all $x \in H$, $y \in G$. (2) follows from (1) and (3) follows from (2) and the Chinese lemma: a direct product of two groups is a cyclic group if and only if they are finite, cyclic of coprime order. \square

2. DEFORMATION OF A MATCHED PAIR

Let H be a group and $\sigma \in \text{Aut}(H)$ an automorphism of H . We define the category $\mathcal{C}(H, \sigma)$ as follows: an object of $\mathcal{C}(H, \sigma)$ is a triple (G, α, β) such that (H, G, α, β) is a matched pair of groups. A morphism $\psi : (G', \alpha', \beta') \rightarrow (G, \alpha, \beta)$ in $\mathcal{C}(H, \sigma)$ is a morphism of groups $\psi : H \rtimes_{\alpha'\rtimes\beta'} G' \rightarrow H \rtimes_{\alpha\rtimes\beta} G$ such that the following diagram

$$\begin{array}{ccc} H & \xrightarrow{i_H} & H \rtimes_{\alpha'\rtimes\beta'} G' \\ \downarrow \sigma & & \downarrow \psi \\ H & \xrightarrow{i_H} & H \rtimes_{\alpha\rtimes\beta} G \end{array} \quad (7)$$

is commutative. A (iso)morphism $\psi : H \rtimes_{\alpha'\rtimes\beta'} G' \rightarrow H \rtimes_{\alpha\rtimes\beta} G$ in the category $\mathcal{C}(H, \sigma)$ will be called a σ -invariant (iso)morphism between the two bicrossed products.

The following key proposition describes explicitly the morphisms of $\mathcal{C}(H, \sigma)$ and gives a necessary and sufficient condition for two bicrossed products $H \rtimes_{\alpha'\rtimes\beta'} G'$ and $H \rtimes_{\alpha\rtimes\beta} G$ to be isomorphic in the category $\mathcal{C}(H, \sigma)$. If G' is a new group we shall denote by "*" the multiplication of G' and $\alpha'(g', h) = g' \triangleright' h$, $\beta'(g', h) = g' \triangleleft' h$, for all $g' \in G'$ and $h \in H$.

Proposition 2.1. *Let H be a group, $\sigma \in \text{Aut}(H)$ and (H, G, α, β) , (H, G', α', β') two matched pairs. There exists a one to one correspondence between the set of all morphisms*

$\psi : H_{\alpha' \bowtie \beta'} G' \rightarrow H_{\alpha \bowtie \beta} G$ in the category $\mathcal{C}(H, \sigma)$ and the set of all pairs (r, v) , where $r : G' \rightarrow H$, $v : G' \rightarrow G$ are two maps such that:

$$\sigma(g' \triangleright' h) r(g' \triangleleft' h) = r(g') (v(g') \triangleright \sigma(h)) \quad (8)$$

$$v(g' \triangleleft' h) = v(g') \triangleleft \sigma(h) \quad (9)$$

$$r(g'_1 * g'_2) = r(g'_1) (v(g'_1) \triangleright r(g'_2)) \quad (10)$$

$$v(g'_1 * g'_2) = (v(g'_1) \triangleleft r(g'_2)) v(g'_2) \quad (11)$$

for all $g', g'_1, g'_2 \in G'$, $h \in H$. Through the above bijection ψ is given by

$$\psi(h, g') = (\sigma(h) r(g'), v(g')) \quad (12)$$

for all $h \in H$, $g' \in G'$. Moreover, $\psi : H_{\alpha' \bowtie \beta'} G' \rightarrow H_{\alpha \bowtie \beta} G$ is an isomorphism in $\mathcal{C}(H, \sigma)$ if and only if the map $v : G' \rightarrow G$ is bijective.

Proof. A morphism of groups $\psi : H_{\alpha' \bowtie \beta'} G' \rightarrow H_{\alpha \bowtie \beta} G$ that makes the diagram (7) commutative is uniquely defined by two maps $r = r_\psi : G' \rightarrow H$, $v = v_\psi : G' \rightarrow G$ such that $\psi(1, g') = (r(g'), v(g'))$ for all $g' \in G'$. In this case ψ is given by :

$$\psi(h, g') = \psi((h, 1) \cdot (1, g')) = (\sigma(h), 1) \cdot (r(g'), v(g')) = (\sigma(h) r(g'), v(g'))$$

for all $h \in H$ and $g' \in G'$. As $\psi(1, 1) = (1, 1)$ we obtain that $r(1) = 1$ and $v(1) = 1$.

We shall prove now that ψ is a morphism of groups if and only if the compatibility conditions (8) - (11) hold for the pair (r, v) . It is enough to check the condition $\psi(xy) = \psi(x)\psi(y)$ only for generators $x, y \in (H \times \{1\}) \cup (\{1\} \times G')$ of the bicrossed product $H_{\alpha' \bowtie \beta'} G'$. Since σ is an automorphism of H , we have to check only for $x = (1, g')$, $y = (h, 1)$ and $x = (1, g'_1)$, $y = (1, g'_2)$. The condition $\psi((1, g')(h, 1)) = \psi(1, g')\psi(h, 1)$ is equivalent to (8) - (9) and the condition $\psi((1, g'_1)(1, g'_2)) = \psi(1, g'_1)\psi(1, g'_2)$ is equivalent to (10) - (11). Note that the normalization conditions $v(1) = 1$ and $r(1) = 1$ were used to obtain (8) and (11).

Conversely, the normalization conditions follow from (8) - (11) in the following manner: first, for $g' = 1$ in (8) we obtain $\sigma(h)r(1) = r(1)(v(1) \triangleright \sigma(h))$ for all $h \in H$. Since σ is an automorphism we have :

$$hr(1) = r(1)(v(1) \triangleright h) \quad (13)$$

for all $h \in H$. Now let $g'_1 = 1$ in (10) to obtain:

$$r(g'_2) = r(1)(v(1) \triangleright r(g'_2)) \stackrel{(13)}{=} r(g'_2)r(1)$$

thus $r(1) = 1$. Finally we let $g'_2 = 1$ in (11) to obtain $v(g'_1) = v(g'_1)v(1)$, thus $v(1) = 1$.

It remains to be proven that ψ given by (12) is an isomorphism if and only if $v : G' \rightarrow G$ is a bijective map. Assume first that ψ is an isomorphism. Then v is surjective and for $g'_1, g'_2 \in G'$ such that $v(g'_1) = v(g'_2)$ we have:

$$\psi(1, g'_2) = (r(g'_2), v(g'_2)) = (r(g'_2), v(g'_1)) = \psi(\sigma^{-1}(r(g'_2))\sigma^{-1}(r(g'_1)^{-1}), g'_1)$$

Hence $g'_1 = g'_2$ and v is injective. Conversely, assume that v is bijective. If $\psi(h, g') = (1, 1)$ we obtain that $\sigma(h)r(g') = 1$ and $v(g') = 1 = v(1)$. It follows from here that $g' = 1$ and $\sigma(h) = 1 = \sigma(1)$, i.e. $h = 1$. Hence ψ is injective. Let $(h, g) \in H_{\alpha \bowtie \beta} G$ and $g' \in G'$

such that $v(g') = g$. Then $\psi(\sigma^{-1}(h)\sigma^{-1}(r(g')^{-1}), g') = (h, g)$ i.e. ψ is an isomorphism of groups. \square

We shall prove now the main result of this section:

Theorem 2.2. (Deformation of a matched pair) *Let (H, G, α, β) be a matched pair of groups, (σ, v, r) be a triple where $\sigma \in \text{Aut}(H)$, $v : G \rightarrow G$ is a bijective map, $r : G \rightarrow H$ is a map such that $v(1_G) = 1_G$, $r(1_G) = 1_H$ and the following compatibility condition:*

$$r \circ v^{-1}((v(g_1) \triangleleft r(g_2))v(g_2)) = r(g_1)(v(g_1) \triangleright r(g_2)) \quad (14)$$

holds for all $g_1, g_2 \in G$. On the set G we define a new multiplication $$ and two new actions $\beta' : G \times H \rightarrow G$, $\alpha' : G \times H \rightarrow H$ given by:*

$$g_1 * g_2 := v^{-1}((v(g_1) \triangleleft r(g_2))v(g_2)) \quad (15)$$

$$g \triangleleft' h := v^{-1}(v(g) \triangleleft \sigma(h)) \quad (16)$$

$$g \triangleright' h := \sigma^{-1}(r(g))\sigma^{-1}(v(g) \triangleright \sigma(h))\sigma^{-1}(r \circ v^{-1}(v(g) \triangleleft \sigma(h))^{-1}) \quad (17)$$

for all $g_1, g_2, g \in G$ and $h \in H$. Then:

- (1) $(G, *)$ is a group structure on the set G with 1_G as a unit;
- (2) $(H, (G, *), \alpha', \beta')$ is a matched pair of groups and

$$\psi : H_{\alpha' \triangleright \beta'}(G, *) \rightarrow H_{\alpha \triangleright \beta} G, \quad \psi(h, g) := (\sigma(h)r(g), v(g))$$

is a σ -invariant isomorphism of groups.

- (3) Any σ -invariant isomorphism of groups $H_{\alpha' \triangleright \beta'} G \cong H_{\alpha \triangleright \beta} G$ arises as above.

Proof. (1) Let $g \in G$. Then

$$g * 1_G = v^{-1}((v(g) \triangleleft r(1_G))v(1)) = v^{-1}(v(g)) = g$$

and

$$1_G * g = v^{-1}((v(1_G) \triangleleft r(g))v(g)) = v^{-1}(v(g)) = g$$

Hence 1_G is a unit for $*$. Let $g_1, g_2, g_3 \in G$. Then:

$$\begin{aligned} v((g_1 * g_2) * g_3) &\stackrel{(15)}{=} v[v^{-1}((v(g_1) \triangleleft r(g_2))v(g_2)) * g_3] \\ &\stackrel{(15)}{=} v[\underline{((v(g_1) \triangleleft r(g_2))v(g_2)) \triangleleft r(g_3)}]v(g_3) \\ &\stackrel{(3)}{=} v[\underline{(v(g_1) \triangleleft r(g_2)) \triangleleft (v(g_2) \triangleright r(g_3))}] (v(g_2) \triangleleft r(g_3))v(g_3) \\ &\stackrel{\triangleleft - \text{ right action}}{=} v[g_1 \triangleleft \underline{(r(g_2)(v(g_2) \triangleright r(g_3)))}] (v(g_2) \triangleleft r(g_3))v(g_3) \\ &\stackrel{(15)}{=} v[g_1 \triangleleft r \circ v^{-1}((v(g_2) \triangleleft r(g_3))v(g_3))] \\ &\quad (v(g_2) \triangleleft r(g_3))v(g_3) \\ &= v[g_1 * \underline{v^{-1}((v(g_2) \triangleleft r(g_3))v(g_3))}] \\ &\stackrel{(15)}{=} v(g_1 * (g_2 * g_3)) \end{aligned}$$

i.e. the multiplication $*$ is associative as v is a bijection. Let $g \in G$ and define $g' := v^{-1}(v(g)^{-1} \triangleleft r(g)^{-1})$. Then:

$$\begin{aligned} v(g' * g) &= (v(g') \triangleleft r(g))v(g) = \left((v(g)^{-1} \triangleleft r(g)^{-1}) \triangleleft r(g) \right) v(g) \\ &= v(g)^{-1}v(g) = 1 = v(1) \end{aligned}$$

i.e. $g' * g = 1$ as v is bijective. Thus every element $g \in G$ has a left inverse, i.e. $(G, *)$ is a group.

(2) The proof can be done directly through a long computation but we prefer the following approach: first we remark that the defining relations (15), (16), (17) are exactly the compatibility conditions (11), (9), and respectively (8) from Proposition 2.1 and the compatibility condition (14) is exactly (10) with the $*$ operations as defined by (15). Moreover the map

$$\psi : H \times (G, *) \rightarrow H \rtimes_{\alpha \bowtie \beta} G, \quad \psi(h, g) = (\sigma(h)r(g), v(g))$$

is a bijection between the set $H \times (G, *)$ and the group $H \rtimes_{\alpha \bowtie \beta} G$. With this observation in mind, in order to prove that $(H, (G, *), \alpha', \beta')$ is a matched pair it is enough to show that the group structure obtained by transferring the group structure from the bicrossed product $H \rtimes_{\alpha \bowtie \beta} G$, to the set $H \times (G, *)$ via the bijective map ψ is exactly the one of a bicrossed product on the set $H \times (G, *)$ associated to the actions α' and β' . In other words, we have to prove that

$$(h(g \triangleright' h'), (g \triangleleft' h')g') = \psi^{-1}(\psi(h, g) \cdot \psi(h', g'))$$

for all $h, h' \in H, g, g' \in G$ or equivalently, as ψ is bijective

$$\psi((h, g) \cdot (h', g')) = \psi(h, g) \cdot \psi(h', g') \quad (18)$$

for all $h, h' \in H, g, g' \in G$. This reduces to proving the following two conditions:

$$\sigma(h(g \triangleright' h'))r \circ v^{-1}[(v(g \triangleleft' h') \triangleleft r(g'))v(g')] = \sigma(h)r(g)(v(g) \triangleright (\sigma(h')r(g'))) \quad (19)$$

and

$$(v(g \triangleleft' h') \triangleleft r(g'))v(g') = (v(g) \triangleleft (\sigma(h')r(g')))v(g') \quad (20)$$

for any $h, h' \in H, g, g' \in G$. We have:

$$\begin{aligned}
\sigma(h)r(g)(v(g) \triangleright (\sigma(h')r(g'))) &\stackrel{(2)}{=} \sigma(h)r(g)(v(g) \triangleright \sigma(h'))((v(g) \triangleleft \sigma(h')) \triangleright r(g')) \\
&\stackrel{(16)}{=} \sigma(h)r(g)(v(g) \triangleright \sigma(h'))(v(g) \triangleleft' h') \triangleright r(g')) \\
&= \sigma(h)r(g)(v(g) \triangleright \sigma(h'))r(g \triangleleft' h')^{-1}r(g \triangleleft' h') \\
&\quad (v(g \triangleleft' h') \triangleright r(g')) \\
&\stackrel{(14)}{=} \sigma(h)r(g)(v(g) \triangleright \sigma(h'))r(g \triangleleft' h')^{-1}r((g \triangleleft' h') * g') \\
&\stackrel{(16),(15)}{=} \sigma(h)r(g)(v(g) \triangleright \sigma(h'))r \circ v^{-1}(v(g) \triangleleft \sigma(h))^{-1} \\
&\quad r \circ v^{-1}[(v(g \triangleleft' h') \triangleleft r(g'))v(g')] \\
&\stackrel{(17)}{=} \sigma(h)\sigma(g \triangleright' h')r \circ v^{-1}[(v(g \triangleleft' h') \triangleleft r(g'))v(g')] \\
&= \sigma(h(g \triangleright' h'))r \circ v^{-1}[(v(g \triangleleft' h') \triangleleft r(g'))v(g')]
\end{aligned}$$

hence (19) holds. Moreover:

$$(v(g \triangleleft' h') \triangleleft r(g')) \stackrel{(16)}{=} [(v(g) \triangleleft \sigma(h')) \triangleleft r(g')]v(g') = (v(g) \triangleleft (\sigma(h')r(g'))v(g'))$$

It follows that (20) and hence (18) holds and we are done.

(3) Follows from (2) and the Proposition 2.1. \square

3. SCHREIER TYPE THEOREMS FOR BICROSSED PRODUCTS

In this section we shall prove two Schreier type classification theorems for bicrossed products. Let H and G be two fixed groups.

Let $MP(H, G) := \{(\alpha, \beta) \mid (H, G, \alpha, \beta) \text{ is a matched pair}\}$. We define $B_1(H, G)$ to be the category having as objects the set $MP(H, G)$ and the morphisms defined as follows: $\psi : (\alpha', \beta') \rightarrow (\alpha, \beta)$ is a morphism in $B_1(H, G)$ if and only if $\psi : H_{\alpha' \bowtie \beta'} G \rightarrow H_{\alpha \bowtie \beta} G$ is a morphism of groups such that $\psi \circ i_H = i_H$, where $i_H(h) = (h, 1)$ is the canonical inclusion. Thus a morphism in the category $B_1(H, G)$ is a morphism between two bicrossed products of H and G that fix H . Considering $\sigma = Id_H$ and $G' = G$ in Proposition 2.1 we obtain the following:

Corollary 3.1. *Let (H, G, α, β) and (H, G, α', β') two matched pairs. There exists a one to one correspondence between the set of all morphisms $\psi : H_{\alpha' \bowtie \beta'} G \rightarrow H_{\alpha \bowtie \beta} G$ in the category $B_1(H, G)$ and the set of all pairs (r, v) , where $r : G \rightarrow H, v : G \rightarrow G$ are two maps such that:*

$$(g \triangleright' h)r(g \triangleleft' h) = r(g)(v(g) \triangleright h) \quad (21)$$

$$v(g \triangleleft' h) = v(g) \triangleleft h \quad (22)$$

$$r(g_1 g_2) = r(g_1)(v(g_1) \triangleright r(g_2)) \quad (23)$$

$$v(g_1 g_2) = (v(g_1) \triangleleft r(g_2))v(g_2) \quad (24)$$

for all $h \in H$, $g, g_1, g_2 \in G$. Through the above bijection ψ is given by

$$\psi(h, g) = (hr(g), v(g)) \quad (25)$$

and $\psi : H_{\alpha' \bowtie_{\beta'}} G \rightarrow H_{\alpha \bowtie_{\beta}} G$ is an isomorphism of groups that fixes H if and only if $v : G \rightarrow G$ is a bijective map.

We are led to the following:

Definition 3.2. Let H and G be two groups. Two pairs $(\alpha, \beta), (\alpha', \beta') \in MP(H, G)$ are called *1-equivalent* and we denote this by $(\alpha, \beta) \approx_1 (\alpha', \beta')$ if and only if there exists a pair (r, v) , where $r : G \rightarrow H$, $v : G \rightarrow G$ are two maps such that v is bijective and the relations (21) - (24) hold.

Thus, using Corollary 3.1, we obtain that $(\alpha, \beta) \approx_1 (\alpha', \beta')$ if and only if there exists an isomorphism $(\alpha, \beta) \cong (\alpha', \beta')$ in the category $B_1(H, G)$. In particular, \approx_1 is an equivalence relation on the set $MP(H, G)$ and we have proved:

Theorem 3.3. (First Schreier type theorem for bicrossed products) Let H and G be two groups. There exists a bijection between the set of objects of the skeleton of the category $B_1(H, G)$ and the pointed quotient set $MP(H, G)/\approx_1$. We shall use the following notation: $K^2(H, G) := M(H, G)/\approx_1$.

A general problem arises in order to have a classification type theorem for bicrossed products of two given groups H and G :

Compute $K^2(H, G)$ for two given groups H and G .

In the last section we shall compute explicitly the set $K^2(C_3, C_6)$. $K^2(H, G)$ is a pointed set by the equivalence class of the pair (α_0, β_0) , where α_0, β_0 are the trivial actions. For $\alpha := \alpha_0$ and $\beta := \beta_0$ we obtain from relations (21) - (24) that \triangleleft' is the trivial action, $r : G \rightarrow H$ and $v : G \rightarrow G$ are morphisms of groups and $g \triangleright' h = r(g)hr(g)^{-1}$, for all $g \in G$ and $h \in H$. For every morphism of groups $r : G \rightarrow H$ we shall denote by \triangleright_r the action $g \triangleright_r h := r(g)hr(g)^{-1}$, for all $g \in G$ and $h \in H$. Hence $(\alpha_0, \beta_0) = \{(\triangleright_r, \beta_0) \mid r : G \rightarrow H \text{ is a morphism of groups}\}$. We restate this observation as follows: let H and G be two groups. Then there exists (H, G, α', β') a matched pair such that $H_{\alpha' \bowtie_{\beta'}} G \cong H \times G$ (isomorphism of groups that fixes H) if and only if the action β' is trivial and there exists a morphism of groups $r : G \rightarrow H$ such that the action α' is given by $g \triangleright' h = r(g)hr(g)^{-1}$ for all $g \in G, h \in H$.

More generally, as a first application of Theorem 3.3, we shall prove the following necessary and sufficient condition for a bicrossed product to be isomorphic to a left version of a semidirect product in the category $B_1(H, G)$:

Corollary 3.4. Let H, G be two groups and $\alpha : G \times H \rightarrow H$ be an action as automorphisms of G on H . The following statements are equivalent:

- (1) There exists (H, G, α', β') a matched pair of groups such that $H_{\alpha' \bowtie_{\beta'}} G \cong H_{\alpha} \ltimes G$ an isomorphism of groups that fixes H .

- (2) The action β' is trivial and there exists a pair (r, v) , where $v \in \text{Aut}(G)$ is an automorphism of G , $r : G \rightarrow H$ is a map such that

$$r(g_1 g_2) = r(g_1)(v(g_1) \triangleright r(g_2)) \quad (26)$$

for all $g_1, g_2 \in G$ and the action α' is given by

$$g \triangleright' h = r(g)(v(g) \triangleright h)r(g)^{-1} \quad (27)$$

for all $g \in G$ and $h \in H$.

The isomorphism $\psi : H_{\alpha'} \bowtie_{\beta'} G \rightarrow H_{\alpha} \ltimes G$ in $B_1(H, G)$ is given by $\psi(h, g) = (hr(g), v(g))$, for all $h \in H, g \in G$.

Proof. We apply Corollary 3.1 in the case that β is the trivial action. In this context, using the fact that v is bijective, it follows from (22) that the action β' is trivial and (24) reduces to the fact that v is a morphism, hence an automorphism of G . Finally, (21) and (23) are exactly (27) and (26). \square

We shall give now a necessary and sufficient condition for a bicrossed product to be isomorphic to a right version of a semidirect product in the category $B_1(H, G)$:

Corollary 3.5. *Let H, G be two groups and $\beta : G \times H \rightarrow G$ be an action as automorphisms of H on G . The following statements are equivalent:*

- (1) *There exists (H, G, α', β') a matched pair of groups such that $H_{\alpha'} \bowtie_{\beta'} G \cong H \rtimes_{\beta} G$ is an isomorphism of groups that fixes H .*
- (2) *There exists a pair (r, v) , where $r : G \rightarrow H$ is a morphism of groups, $v : G \rightarrow G$ is a bijective map such that*

$$v(g_1 g_2) = (v(g_1) \triangleleft r(g_2))v(g_2) \quad (28)$$

for all $g_1, g_2 \in G$ and the actions α' and β' are given by

$$g \triangleright' h = r(g)h \left(r \circ v^{-1}(v(g) \triangleleft h) \right)^{-1} \quad (29)$$

$$g \triangleleft' h = v^{-1}(v(g) \triangleleft h) \quad (30)$$

for all $g \in G$ and $h \in H$.

The isomorphism $\psi : H_{\alpha'} \bowtie_{\beta'} G \rightarrow H \rtimes_{\beta} G$ in $B_1(H, G)$ is given by $\psi(h, g) = (hr(g), v(g))$, for all $h \in H, g \in G$.

Proof. We apply Corollary 3.1 in the case that α is the trivial action. In this context, it follows from (23) that r is a morphism of groups, while (21) and (22) are exactly (29) and (30). \square

In what follows we will prove the second Schreier type theorem for bicrossed products: it is the analogue of the theorem regarding group extensions. Let (H, G, α, β) be a matched pair. Then the natural projections $\pi_G : H_{\alpha} \bowtie_{\beta} G \rightarrow G$, $\pi_H : H_{\alpha} \bowtie_{\beta} G \rightarrow H$ are not morphisms of groups.

We will fix two groups H, G and $\beta : G \times H \rightarrow G$ a right action of the group H on the set G . We define

$$\text{Ker}(\beta) := \{h \in H \mid g \triangleleft h = g, \forall g \in G\}$$

We denote by $MP_\beta(H, G) := \{\alpha \mid (H, G, \alpha, \beta) \text{ is a matched pair}\}$. Let $B_2^\beta(H, G)$ be the category having $MP_\beta(H, G)$ as the set of objects and the morphisms defined as follows: $\psi : \alpha' \rightarrow \alpha$ is a morphism in $B_2^\beta(H, G)$ if and only if $\psi : H_{\alpha' \bowtie_\beta} G \rightarrow H_{\alpha \bowtie_\beta} G$ is a morphism of groups such that

$$\psi \circ i_H = i_H \text{ and } \pi_G \circ \psi = \pi_G \quad (31)$$

Proposition 3.6. *Let $(H, G, \alpha', \beta), (H, G, \alpha, \beta)$ be two matched pairs. There exists a one to one correspondence between the set of all morphisms $\psi : \alpha' \rightarrow \alpha$ in the category $B_2^\beta(H, G)$ and the set of all maps $r : G \rightarrow \text{Ker}(\beta)$ such that :*

$$(g \triangleright' h)r(g \triangleleft h) = r(g)(g \triangleright h) \quad (32)$$

$$r(g_1 g_2) = r(g_1)(g_1 \triangleright r(g_2)) \quad (33)$$

for all $g, g_1, g_2 \in G, h \in H$. Through the above bijection the morphism ψ is given by

$$\psi(h, g) = (hr(g), g) \quad (34)$$

for all $h \in H, g \in G$ and ψ is an isomorphism of groups i.e. $B_2^\beta(H, G)$ is a grupoid¹.

Proof. For any morphism of groups $\psi : H_{\alpha' \bowtie_\beta} G \rightarrow H_{\alpha \bowtie_\beta} G$ such that (31) hold there exists a unique map $r : G \rightarrow H$ such that $\psi(h, g) = (hr(g), g)$, for all $h \in H$ and $g \in G$. Now we are in a position to use Proposition 2.1 for $G' = G, \beta' = \beta$ and $v = Id_G$. We obtain (32) and (33) by considering $v = Id_G$ in (8), respectively (10). On the other hand (9) is trivially fulfilled and (11) becomes $g_1 = g_1 \triangleleft r(g_2)$ for all $g_1, g_2 \in G$, i.e. $\text{Im}(r) \subseteq \text{Ker}(\beta)$. \square

Remark 3.7. If β is a faithful action (i.e. $\text{Ker}(\beta) = 1$) then $B_2^\beta(H, G)$ is a discret grupoid i.e. there exists a morphism $\psi : \alpha' \rightarrow \alpha$ if and only if $\alpha = \alpha'$. Indeed, in this case $r(g) = 1$ for all $g \in G$, (33) is trivially fulfilled and (32) reduces to $g \triangleright h = g \triangleright' h$ i.e. $\alpha = \alpha'$. Hence, in this case the skeleton of the category $B_2^\beta(H, G)$ is the set $MP_\beta(H, G)$.

Definition 3.8. Let H, G be two groups and $\beta : G \times H \rightarrow G$ be a right action. Two elements α' and α of $MP_\beta(H, G)$ are called \approx_2 -equivalent and we denote this by $\alpha' \approx_2 \alpha$ if there exists a map $r : G \rightarrow \text{Ker}(\beta)$ such that the relations (32) and (33) hold.

From Proposition 3.6 we obtain that $\alpha' \approx_2 \alpha$ if and only if there exists an isomorphism $\alpha' \cong \alpha$ in $B_2^\beta(H, G)$. Hence \approx_2 is an equivalence relation on $MP_\beta(H, G)$ and we obtained the following:

Theorem 3.9. (The second Schreier type theorem for bicrossed products) *Let H, G be two groups and $\beta : G \times H \rightarrow G$ be a right action. There exists a bijection between the set of objects of the skeleton of the category $B_2^\beta(H, G)$ and the quotient set $MP_\beta(H, G)/\approx_2$.*

¹We recall that a grupoid is a category such that the class of objects is a set and any morphism is an isomorphism.

It is possible that the set $MP_\beta(H, G)$ (and hence $MP_\beta(H, G)/\approx_2$) is the empty set. However, if $\beta : G \times H \rightarrow G$ is an action as automorphisms then $MP_\beta(H, G)$ is nonempty as it contains the trivial action α_0 . In this case the quotient set $MP_\beta(H, G)/\approx_2$ is a pointed set by the equivalence class of the trivial action α_0 . It follows from Proposition 3.6 that

$$\widehat{\alpha}_0 = \{\alpha' \mid \alpha'(g, h) = r(g)hr(g \triangleleft h)^{-1}, \text{ for some } r : G \rightarrow \text{Ker}(\beta) \text{ a morphism of groups}\}$$

We record this observation in the following:

Corollary 3.10. *Let H, G be two groups, $\beta : G \times H \rightarrow G$ an action as automorphisms and $H \rtimes_\beta G$ the right version of the semidirect product. The following statements are equivalent:*

- (1) *There exists a matched pair (H, G, α, β) such that the bicrossed products $H \rtimes_{\alpha \rtimes \beta} G$ and $H \rtimes_\beta G$ are isomorphic in the category $B_2^\beta(H, G)$;*
- (2) *There exists a morphism of groups $r : G \rightarrow \text{Ker}(\beta)$ such that the action α is given by $g \triangleright h = r(g)hr(g \triangleleft h)^{-1}$ for all $g \in G, h \in H$.*

4. EXAMPLES

In this section we describe all matched pairs between C_n and C_m , for $n \in \{2, 3\}$ and $m \in \mathbb{N}^*$ arbitrary. First, let us introduce some notation. We denote by a a generator of the cyclic group C_n and b a generator of C_m . The set of group morphisms from the group C_n to the group of automorphisms $\text{Aut}(C_m)$ will be denoted by $\varsigma(n, m)$. Such a morphism $\vartheta : C_n \rightarrow \text{Aut}(C_m)$ is uniquely determined by a positive integer $t \in [m-1] := \{1, 2, \dots, m-1\}$ such that $m \mid t^n - 1$ and

$$\vartheta : C_n \rightarrow \text{Aut}(C_m), \quad \vartheta(a)(b) = b^t \tag{35}$$

Therefore, one can equivalently think of $\varsigma(n, m)$ as the subgroup of $U(\mathbb{Z}_m)$ consisting of all solutions in \mathbb{Z}_m of the equation $x^n = 1$.

Using the fact that if $m = 2^{a_0} p_1^{a_1} \cdots p_k^{a_k}$ with p_1, \dots, p_k odd primes, then

$$\text{Aut}(C_m) \cong U(\mathbb{Z}_m) \cong U(\mathbb{Z}_{2^{a_0}}) \times U(\mathbb{Z}_{p_1^{a_1}}) \times \cdots \times U(\mathbb{Z}_{p_k^{a_k}})$$

it is a routine computation to check that

$$|\varsigma(n, m)| = \begin{cases} \prod_{i=1}^k (n, p_i^{a_i} - p_i^{a_i-1}), & \text{if } 4 \nmid m \\ (n, 2)(n, 2^{a_0-2}) \prod_{i=1}^k (n, p_i^{a_i} - p_i^{a_i-1}), & \text{if } 4 \mid m \end{cases}$$

In particular

$$|\varsigma(2, m)| = \begin{cases} 2^k, & \text{if } a_0 \leq 1 \\ 2^{k+1}, & \text{if } a_0 = 2 \\ 2^{k+2}, & \text{if } a_0 \geq 3 \end{cases}$$

and

$$|\varsigma(p, m)| = \begin{cases} \prod_{i=1}^k (p, p_i - 1), & \text{if } p^2 \nmid m \\ p \prod_{i=1}^k (p, p_i - 1), & \text{if } p^2 \mid m \end{cases}$$

for an odd prime p .

Let m be a positive integer. Then $(C_2, C_m, \alpha, \beta)$ is a matched pair if and only if the action α is trivial and there exists a positive integer $t \in [m-1]$ such that $m|t^2 - 1$ and $\beta = \beta_t : C_m \times C_2 \rightarrow C_m$ is given by

$$\beta(b^i, a) = b^{it}, \quad \beta(b^i, 1) = b^i \quad (36)$$

for any $i = 0, \dots, m-1$. In particular, there are $|\varsigma(2, m)|$ matched pairs $(C_2, C_m, \alpha, \beta)$. Indeed, as α is an action we get $b \triangleright a \neq 1 = b \triangleright 1$. Thus $b \triangleright a = a$ which implies that α is trivial. Thus $(C_2, C_m, \alpha, \beta)$ is a matched pair if and only if $\beta' : C_2 \rightarrow \text{Aut}(C_m)$, $\beta'(x)(y) := \beta(y, x)$ is a morphism of groups, so by letting $n = 2$ in (35) we obtain that $(C_2, C_m, \alpha, \beta)$ is a matched pair if and only if there exists $t \in [m-1]$ such that $m|t^2 - 1$ and $\beta(b, a) = b^t$. The formula (36) follows as β is an action.

In order to describe all matched pairs $(C_3, C_m, \alpha, \beta)$ we need the following observation.

Remark 4.1. Let (H, G, α, β) be a matched pair such that α is an action of G on H as group automorphisms. Then the compatibility condition (2) from the definition of a matched pair is equivalent to $(g \triangleleft h_1) \triangleright h_2 = g \triangleright h_2$ that can be written as

$$g^{-1}(g \triangleleft h_1) \in \text{Stab}_G(h_2) \quad (37)$$

for any $g \in G$, $h_1, h_2 \in H$. Thus if α is an action as automorphisms then (H, G, α, β) is a matched pair if and only if (3) and (37) hold. The condition (37) gives important information regarding β : the elements $g^{-1}\beta(g, h)$ act trivially on H for any $g \in G$ and $h \in H$.

Now we can describe all matched pairs $(C_3, C_m, \alpha, \beta)$.

Proposition 4.2. *Let m be a positive integer, $\alpha : C_m \times C_3 \rightarrow C_3$, $\beta : C_m \times C_3 \rightarrow C_m$ two maps and $t \in [m-1]$ such that $m|t^3 - 1$. Then:*

- (i) *Let α be the trivial action and $\beta = \beta_t : C_m \times C_3 \rightarrow C_m$ given by*

$$\beta(b^i, a) = b^{it}, \quad \beta(b^i, a^2) = b^{it^2}, \quad \beta(b^i, 1) = b^i \quad (38)$$

for any $i = 0, \dots, m-1$. Then $(C_3, C_m, \alpha, \beta_t)$ is a matched pair. There are no other matched pairs $(C_3, C_m, \alpha, \beta)$ if m is odd.

- (ii) *Assume that m is even. Let β be the trivial action and $\alpha : C_m \times C_3 \rightarrow C_3$ given by $\alpha(b^j, 1) = 1$ and*

$$\alpha(b^j, a) = \begin{cases} a, & \text{if } j \text{ is even} \\ a^2, & \text{if } j \text{ is odd} \end{cases} \quad (39)$$

$$\alpha(b^j, a^2) = \begin{cases} a^2, & \text{if } j \text{ is even} \\ a, & \text{if } j \text{ is odd} \end{cases} \quad (40)$$

for all $j = 1, \dots, m-1$. Then $(C_3, C_m, \alpha, \beta)$ is a matched pair.

- (iii) *Assume that $m = 6u$ for some positive integer u and that α is described by (39) and (40). Then there exist two matched pairs $(C_3, C_m, \alpha, \beta)$, $(C_3, C_m, \alpha, \beta')$, where β and β' are given by*

$$\beta(b^{2k+1}, a) = b^{2u+2k+1}, \quad \beta(b^{2k+1}, a^2) = b^{4u+2k+1} \quad (41)$$

$$\beta(b^{2k}, a) = \beta(b^{2k}, a^2) = b^{2k} \quad (42)$$

and

$$\beta'(b^{2k+1}, a) = b^{4u+2k+1}, \quad \beta'(b^{2k+1}, a^2) = b^{2u+2k+1} \quad (43)$$

$$\beta'(b^{2k}, a) = \beta'(b^{2k}, a^2) = b^{2k} \quad (44)$$

for all nonnegative integers k . In this case there are $2 + |\varsigma(3, m)|$ matched pairs between C_3 and C_m .

- (iv) There are no other matched pairs on $(C_3, C_m, \alpha, \beta)$ other than the ones described above.

Proof. We assume first that α is the trivial action. It follows from Remark 1.1 that $(C_3, C_m, \alpha, \beta)$ is a matched pair if and only if $\beta' : C_3 \rightarrow \text{Aut}(C_m)$, $\beta'(x)(y) := \beta(y, x)$ is a morphism of groups; setting $n = 3$ in (35) we get that $(C_3, C_m, \alpha, \beta)$ is a matched pair if and only if there exists $t \in [m - 1]$ such that $m \nmid t^3 - 1$ and $\beta(b, a) = b^t$. The formula (38) follows as β is an action.

Assume now that m is odd. It follows from (4) that $b^i \triangleright 1 = 1$ for all $i = 0, \dots, m - 1$. If $b \triangleright a = a$ we obtain that α is trivial. Assume that $b \triangleright a = a^2$. Then $b \triangleright a^2 = a$ and α is given by (39), (40). If m is odd we obtain: $a = 1 \triangleright a = b^m \triangleright a = a^2$, contradiction. Hence, for an odd m the action α must be trivial and (i) is proved.

Assume now that m is even and β is the trivial action. Then $(C_3, C_m, \alpha, \beta)$ is a matched pair if and only if α is an action of C_m on C_3 as group automorphisms. The map α given by (39), (40) is such an action corresponding to

$$C_m \rightarrow \text{Aut}(C_3), \quad b \mapsto (a \mapsto a^2)$$

Therefore (ii) is proved.

We shall prove now (iii) and (iv). Let $(C_3, C_m, \alpha, \beta)$ be a matched pair. We have proved that $\alpha : C_m \times C_3 \rightarrow C_3$ is either the trivial action or it is given by (39), (40) if m is even.

We assume now that m is even and that α is given by (39), (40). Then α is an action of C_m on C_3 as automorphisms and

$$\text{Stab}_{C_m}(a) = \text{Stab}_{C_m}(a^2) = \langle b^2 \rangle$$

Using Remark 4.1 we get that $b^{-1}\beta(b, a) \in \langle b^2 \rangle$ and $b^{-1}\beta(b, a^2) \in \langle b^2 \rangle$ and (2) holds automatically. Let $l, t \in \{0, 1, \dots, m/2 - 1\}$ such that

$$\beta(b, a) = b^{2l+1}, \quad \beta(b, a^2) = b^{2t+1} \quad (45)$$

We shall extend β for each element of $C_m \times C_3$ using (3) as defining relations and the fact that β is an action. First we define β for each pair (b^i, a) such that (3) holds. We have

$$\beta(b^2, a) = (bb) \triangleleft a \stackrel{(3)}{=} (b \triangleleft (b \triangleright a))(b \triangleleft a) \stackrel{(39)}{=} (b \triangleleft a^2)(b \triangleleft a) = b^{2(t+l+1)}$$

and

$$\beta(b^3, a) = (bb^2) \triangleleft a \stackrel{(3)}{=} (b \triangleleft (b^2 \triangleright a))(b^2 \triangleleft a) \stackrel{(39)}{=} (b \triangleleft a)(b^2 \triangleleft a) = b^{4l+2t+3}$$

Using the induction we can prove

$$\beta(b^{2k}, a) = b^{2k(l+t+1)}, \quad \beta(b^{2k+1}, a) = b^{(2k+2)l+2kt+2k+1} \quad (46)$$

for any $k = 0, 1, \dots$. We note that

$$b^{2t+1} = \beta(b, a^2) = b \triangleleft a^2 = (b \triangleleft a) \triangleleft a = \beta(b^{2l+1}, a) \stackrel{(46)}{=} b^{(2l+2)l+2lt+2l+1}$$

As the order of b is m we get a first compatibility condition for l and t :

$$m|2(l^2 + 2l + lt - t) \quad (47)$$

Now we define β for each pair (b^i, a^2) using (3) repeatedly. We have:

$$\begin{aligned} \beta(b^2, a^2) &= (bb) \triangleleft a^2 \stackrel{(3)}{=} (b \triangleleft (b \triangleright a^2))(b \triangleleft a^2) \stackrel{(40)}{=} (b \triangleleft a)(b \triangleleft a^2) = b^{2(l+t+1)} \\ \beta(b^3, a^2) &= (bb^2) \triangleleft a^2 \stackrel{(3)}{=} (b \triangleleft (b^2 \triangleright a^2))(b^2 \triangleleft a^2) \stackrel{(40)}{=} (b \triangleleft a^2)(b^2 \triangleleft a^2) = b^{2l+4t+3} \end{aligned}$$

Using the induction we can easily prove that

$$\beta(b^{2k}, a^2) = b^{2k(l+t+1)}, \quad \beta(b^{2k+1}, a^2) = b^{2kl+(2k+2)t+2k+1} \quad (48)$$

for any $k = 0, 1, \dots$. Moreover, keeping in mind (46) we find that

$$\beta(b^{2k}, a) = \beta(b^{2k}, a^2) = b^{2k(l+t+1)}$$

On the other hand β is a right action and $a^3 = 1$. Hence:

$$b^{2k} = \beta(b^{2k}, 1) = (b^{2k} \triangleleft a^2) \triangleleft a = (b^{2k} \triangleleft a) \triangleleft a = b^{2k} \triangleleft a^2 = b^{2k(l+t+1)}$$

As the order of b is m we obtain a second compatibility condition between l and t : $m|2k(l+t)$ for any $k = 0, 1, \dots$ which is equivalent to:

$$m|2(l+t) \quad (49)$$

From this condition and (47) we obtain

$$m|2(2l-t) \quad (50)$$

Let now $m = 2r$. We have to find $l, t \in \{1, 2, \dots, r-1\}$ such that

$$m|2(l+t) \quad \text{and} \quad m|2(2l-t)$$

Equivalently, we have to solve in \mathbb{Z}_r the system of equations

$$\begin{cases} \hat{l} + \hat{t} = \hat{0} \\ 2\hat{l} - \hat{t} = \hat{0} \end{cases} \quad (51)$$

The equation $3\hat{l} = \hat{0}$ has $(3, r)$ solutions in \mathbb{Z}_r . If 3 does not divide m then the unique solution of the system is $\hat{l} = \hat{t} = \hat{0}$ and therefore β is the trivial action. If 3 divides r let u be such that $r = 3u$. Then the system (51) has three solutions

$$\hat{l}_1 = \hat{t}_1 = \hat{0}; \quad \hat{l}_2 = \hat{u}, \hat{t}_2 = 2\hat{u}, \quad \hat{l}_3 = 2\hat{u}, \hat{t}_3 = 4\hat{u}$$

The first solution gives that the action β is trivial and the last two solutions give exactly the two actions β described in (41) and (43). \square

We showed that the smallest example of a proper matched pair (i.e. one in which both actions are nontrivial) between two finite cyclic groups is the one between the groups C_3 and C_6 . According to Proposition 4.2 there exist exactly four matched pairs $(C_3, C_6, \alpha, \beta)$ namely:

- (i) α_0 and β_0 are the trivial actions;
- (ii) β_0 is the trivial action and α_1 is defined by:

$$b^j \triangleright_1 a = \begin{cases} a, & \text{if } j \text{ is even} \\ a^2, & \text{if } j \text{ is odd} \end{cases}, \quad b^j \triangleright_1 a^2 = \begin{cases} a^2, & \text{if } j \text{ is even} \\ a, & \text{if } j \text{ is odd} \end{cases}$$

for all $j = 1, \dots, 5$.

- (iii) α_2 and β_2 are defined by :

$$b^j \triangleright_2 a = \begin{cases} a, & \text{if } j \text{ is even} \\ a^2, & \text{if } j \text{ is odd} \end{cases}, \quad b^j \triangleright_2 a^2 = \begin{cases} a^2, & \text{if } j \text{ is even} \\ a, & \text{if } j \text{ is odd} \end{cases}$$

and

$$b^j \triangleleft_2 a = \begin{cases} b^j, & \text{if } j \text{ is even} \\ b^{j+2}, & \text{if } j \text{ is odd} \end{cases}, \quad b^j \triangleleft_2 a^2 = \begin{cases} b^j, & \text{if } j \text{ is even} \\ b^{j+4}, & \text{if } j \text{ is odd} \end{cases}$$

for all $j = 1, \dots, 5$.

- (iv) α_2 and β_3 are defined by:

$$b^j \triangleleft_3 a = \begin{cases} b^j, & \text{if } j \text{ is even} \\ b^{j+4}, & \text{if } j \text{ is odd} \end{cases}, \quad b^j \triangleleft_3 a^2 = \begin{cases} b^j, & \text{if } j \text{ is even} \\ b^{j+2}, & \text{if } j \text{ is odd} \end{cases}$$

for all $j = 1, \dots, 5$.

We shall now classify all bicrossed products $C_3 \bowtie C_6$ that fixes the group C_3 , i.e. we shall determine the pointed set $K^2(C_3, C_6)$ from Theorem 3.3.

Corollary 4.3. *$K^2(C_3, C_6)$ is a pointed set with three elements. In particular, any bicrossed product $C_3 \bowtie C_6$ that fixes the group C_3 is isomorphic to one of the following three groups:*

$$C_3 \times C_6, \quad \langle a, b \mid a^3 = 1, b^6 = 1, ba = a^2b \rangle, \quad \langle a, b \mid a^3 = 1, b^6 = 1, ba = a^2b^3 \rangle$$

Proof. Let (α', β') be a matched pair such that $(\alpha_0, \beta_0) \approx_1 (\alpha', \beta')$. The relations (21) - (22) collapse into:

$$(g \triangleright' h)r(g \triangleleft' h) = r(g)h \tag{52}$$

$$v(g \triangleleft' h) = v(g) \tag{53}$$

Since v is a bijective map, it follows from (53) that β' is the trivial action. Furthermore, from (52) we obtain that α' is also the trivial action, that is, the equivalence class of (α_0, β_0) is trivial. By similar arguments it follows that the equivalence class of (α_1, β_0) is also trivial.

Consider now $r : C_6 \rightarrow C_3$ be the trivial morphism of groups and $v : C_6 \rightarrow C_6$ the automorphism given by $v(b) := b^5$. By a straightforward computation it follows that:

$$g \triangleright_2 h = v(g) \triangleright_2 h, \quad v(g \triangleleft_3 h) = v(g) \triangleleft_2 h$$

hence $(\alpha_2, \beta_2) \approx_1 (\alpha_2, \beta_3)$. Thus, $K^2(C_3, C_6)$ is a set with three elements. \square

Remark 4.4. It is easy to see that $B_2^\beta(C_3, C_6)$ is a singleton or a set with two elements for any right action β .

Indeed, it is obvious that $B_2^{\beta_2}(C_3, C_6) = \{(\alpha_2, \beta_2)\}$ and $B_2^{\beta_3}(C_3, C_6) = \{(\alpha_2, \beta_3)\}$. Now suppose that $(\alpha_0, \beta_0) \approx_2 (\alpha_1, \beta_0)$. From (32) we obtain that $g \triangleright_0 h = g \triangleright_1 h$ for all $g \in C_3, h \in C_6$ which is a contradiction. Thus $B_2^{\beta_0}(C_3, C_6) = \{(\alpha_0, \beta_0), (\alpha_1, \beta_0)\}$.

We end the section by showing the difficulty of the problem of finding all matched pairs $(C_n, C_m, \alpha, \beta)$ between arbitrary finite cyclic groups. For any positive integer k we denote by \mathbb{Z}_k the ring of residue classes modulo k and by $S(\mathbb{Z}_k)$ the set of bijective functions from \mathbb{Z}_k to itself.

Let $\alpha : C_m \times C_n \rightarrow C_n$, and $\beta : C_m \times C_n \rightarrow C_m$ be two actions. They are completely determined by two maps

$$\theta : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \quad \text{and} \quad \phi : \mathbb{Z}_m \rightarrow \mathbb{Z}_m$$

such that

$$\alpha(b, a^x) = a^{\theta(x)}, \quad \beta(b^y, a) = b^{\phi(y)}$$

for any $x \in \mathbb{Z}_n, y \in \mathbb{Z}_m$. As α and β are actions we obtain that θ and ϕ are bijections (and hence they can be regarded as elements of $S(\mathbb{Z}_n)$ and respectively $S(\mathbb{Z}_m)$) and

$$\alpha(b^i, a^x) = a^{\theta^i(x)}, \quad \beta(b^y, a^j) = b^{\phi^j(y)} \quad (54)$$

for all positive integers i , and j . In particular,

$$\theta^m(x) = x \quad \text{and} \quad \phi^n(y) = y \quad (55)$$

for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$. Hence, as an element of $S(\mathbb{Z}_n)$ the order of θ is a divisor of m and as an element of $S(\mathbb{Z}_m)$ the order of ϕ is a divisor of n . Using (54) we obtain immediately that the compatibility conditions (2), (3) defining a matched pair are equivalent to

$$\begin{aligned} \theta^y(x+z) - \theta^y(z) &= \theta^{\phi^z(y)}(x) \\ \phi^y(x+z) - \phi^y(z) &= \phi^{\theta^z(y)}(x) \end{aligned}$$

for all x, y, z and all operations are in the relevant rings. Finally, (4) is equivalent to $\theta(0) = 0$ and $\phi(0) = 0$. To conclude, we have the following

Proposition 4.5. *Let m and n be two positive integers. There is a one to one correspondence between the set of all matched pairs $(C_n, C_m, \alpha, \beta)$ and the set of pairs $(\theta, \phi) \in S(\mathbb{Z}_n) \times S(\mathbb{Z}_m)$ such that*

$$\theta(0) = 0, \quad \phi(0) = 0 \quad (56)$$

$$\text{ord}(\theta) | m, \quad \text{ord}(\phi) | n \quad (57)$$

$$\theta^y(x+z) - \theta^y(z) = \theta^{\phi^z(y)}(x) \quad (58)$$

$$\phi^y(x+z) - \phi^y(z) = \phi^{\theta^z(y)}(x) \quad (59)$$

for all x, y, z (all the operations are taking place in the relevant rings).

This is [8, Theorem I] and the pairs $(\theta, \phi) \in S(\mathbb{Z}_n) \times S(\mathbb{Z}_m)$ satisfying the above conditions are called in [8] *conjugate special substitutions*. Finding all pairs of conjugate special substitutions is still an open question.

REFERENCES

- [1] Agore, A.L., Chirvasitu, A., Ion, B., Militaru, G. Bicrossed products for finite groups. *Algebr. Represent. Theory* (in press), ArXiv: 0703471.
- [2] Aguiar, M., Andruskiewitsch, N. Representations of matched pairs of groupoids and applications to weak Hopf algebras. Algebraic structures and their representations, *Contemp. Math.*, 376 (2005), 127–173, Amer. Math. Soc., Providence.
- [3] Baaj, S., Skandalis, G., Vaes, S. Measurable Kac cohomology for bicrossed products, *Trans. Amer. Math. Soc.*, **357** (2005), 1497–1524.
- [4] Baumeister, B. Factorizations of Primitive Permutation Groups, *J. Algebra* **194** (1997), 631 - 653.
- [5] Caenepeel, S., Ion, B., Militaru, G., and Zhu, S. The factorization problem and the smash biproduct of algebras and coalgebras. *Algebr. Represent. Theory* **3** (2000), no. 1, 19–42.
- [6] Cap, A., Schichl, H., Vanura, J. On twisted tensor products of algebras, *Comm. Algebra* **23** (1995), no. 12, 4701–4735.
- [7] Cohn, P.M., A remark on the general product of two infinite Cyclic groups, *Arch. Math. (Basel)* **7**(1956), 94–99.
- [8] Douglas, J. On finite groups with two independent generators. I, II, III, IV. *Proc. Nat. Acad. Sci. U. S. A.* **37** (1951), 604–610, 677–691, 749–760, 808–813.
- [9] Giudici, M. Factorisations of sporadic simple groups, *J. Algebra* **304** (2006), 311–323.
- [10] Guccione, J. A., Guccione, J.J., Valqui, C. "Twisted planes", ArXiv: 07124094
- [11] Jara, P., López Pena, J., Panaite, F., Van Oystaeyen, F. Invariance under twisting, *New techniques in Hopf algebras and graded ring theory*, 85–104, K. Vlaam. Acad. Belgie Wet. Kunsten (KVAB), Brussels, 2007.
- [12] López Peña, J. and Navarro, G. On the classification and properties of noncommutative duplicates. *K-Theory* **38** (2008), no. 2, 223–234.
- [13] Kroetz, B. A novel characterization of the Iwasawa decomposition of a simple Lie group. *Springer Lecture Notes in Physics*, **723**(2007), 195–201.
- [14] Maillet, E. Sur les groupes échangeables et les groupes décomposables. *Bull. Soc. Math. France* **28** (1900), 7–16.
- [15] Masuoka, A. Hopf algebra extensions and cohomology. *New directions in Hopf algebras*, 167–209, Math. Sci. Res. Inst. Publ., 43, Cambridge Univ. Press, Cambridge, 2002.
- [16] Michor, P. W. Knit products of graded Lie algebras and groups, Proceedings of the Winter School on Geometry and Physics (Srn, 1989). *Rend. Circ. Mat. Palermo* (2) Suppl. No. **22** (1990), 171–175.
- [17] Ore, O. Structures and group theory. I. *Duke Math. J.* **3** (1937), no. 2, 149–174.
- [18] Praeger, C. E. and Schneider, C. Factorisations of characteristically simple groups, *J. Algebra* **255** (2002), 198 - 220.
- [19] Rédei, L. Zur Theorie der faktorisierbaren Gruppen I, *Acta Math. Acad. Sci. Hung.*, **1**(1950), 74–98.
- [20] Takeuchi, M. Matched pairs of groups and bismash products of Hopf algebras, *Comm. Algebra* **9** (1981), no. 8, 841–882.
- [21] Vaes, S. and Vainerman, L. Extensions of locally compact quantum groups and the bicrossed product construction, *Adv. Math.* **175** (2003), no. 1, 1–101.
- [22] Wiegold, J. and Williamson, A. G. The factorisation of the alternating and symmetric groups. *Math. Z.* **175** (1980), no. 2, 171–179.

(A.L.A., G.M.) FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BUCHAREST, STR. ACADEMIEI 14, RO-010014 BUCHAREST 1, ROMANIA

E-mail address: ana.agore@fmi.unibuc.ro

E-mail address: gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com